Errata for

*An Introduction to Manifolds*, Second Edition

Loring W. Tu

June 14, 2020

- p. 6, Proof of Lemma 1.4: For clarity, the point should be called \( y \), instead of \( x \). Use \( x \) only for the argument of \( f \). Thus, in the first three lines of the proof, change the three instances of \( x \) to \( y \). In Figure 1.3, change the two instances of \( x \) to \( y \). Add to the beginning of the second paragraph “By the chain rule, ...” the sentences

Let \( x^1, \ldots, x^n \) be the variables of \( f \). Then in \( f(p + t(y - p)) \),

\[
x^i = p^i + t(y^i - p^i).
\]

In the rest of the proof, change the twelve instances of \( x \) to \( y \), but of course \( \partial f/\partial x^i \) should not be changed.

- p. 9, Problem 1.5(b): Add at the end “(Hint: To show that a map is \( C^\infty \), you may use the fact that the sum, product, quotient, and composition of \( C^\infty \) functions are \( C^\infty \) whenever they are defined.)”

- p. 9, Problem 1.6: Replace \( g_{12} \) by \( h_{12} \) in two places. In the solution on p. 368, replace “gives the result” by “and setting \( h_{12} = g_{12} + g_{21} \) give the result”.

- p. 12, insert after the paragraph defining an algebra:

\[ \text{Example. The set } C^\infty(U) \text{ of all } C^\infty \text{ functions on an open set } U \subset \mathbb{R}^n \text{ is an algebra over } \mathbb{R}. \]

- p. 20, line 5: Delete parentheses around \( a_r \) in its first occurrence.

- p. 20, line 6 of the Example 3.4: “4 → 1” should be “4\( \mapsto \)1”.

- p. 27, Remove the * after Example 3.19 and place it after Exercise 3.20.

- p. 31, proof of Lemma 3.28: Replace the second displayed equation by

\[
\begin{align*}
&i_1 < i_2 < \cdots < i_{\ell - 1} < i_{\ell} \\
&j_1 < j_2 < \cdots < j_{\ell - 1} < j_{\ell} < j_{\ell + 1} < \cdots.
\end{align*}
\]

- p. 31, lines -1, -2, and -4 in the proof of Lemma 3.28: Replace \( a \) by \( \alpha \) in “\( \det[a^j(e_j)] = 0 \)” and “the matrix \([a^j(e_j)]\)”. Also “\( i_1, \ldots, i_{\ell - 1} \)” should be “\( i_1, \ldots, i_{\ell - 1} \)”.

- p. 32, Problem 3.3, line 3: \( A_k(L) \) should be \( A_k(V) \).

- p. 33, Problem 3.9, line 2: zero covector → zero \( n \)-covector.

- p. 37, display -1: Replace \( \omega(X)_p \) by \( \omega(X)(p) \).

- p. 37, insert between display -1 and “Written out in ...”:

This function \( \omega(X) \) is linear in \( X \) over the ring \( C^\infty(U) \); i.e., if \( f \in C^\infty(U) \), then \( \omega(fX) = f\omega(X) \). To show this, it suffices to evaluate \( \omega(fX) \) at an arbitrary point \( p \in U \):

\[
(\omega(fX))(p) = \omega_p(f(p)X_p) \quad \text{(definition of } \omega(fX))
\]
\[
= f(p)\omega_p(X_p) \quad \text{(} \omega_p \text{ is } \mathbb{R} \text{-linear)}
\]
\[
= (f\omega(X))(p) \quad \text{(definition of } f\omega(X)).
\]
• p. 38, delete the second paragraph starting with “This function is actually ...”.

• p. 38, Exercise 4.4, line 2: $M$ should be $\mathbb{R}^3$.

• p. 47, line −2: Replace “finds” by “found”.

• p. 53, Proposition 5.10, lines 1–2 of proof: “Proposition 5.8” should be “Lemma 5.8”.

• p. 54, line 11: $f : U \to \mathbb{R}^n$ should be $f : U \to \mathbb{R}^m$.

• pp. 56–57, Remark: This remark uses the concept of a diffeomorphism, which is not defined until the next section. Move the entire remark consisting of four paragraphs to p. 63, right before Section 6.4.

• p. 61, Definition 6.5, line 3: Insert “with $F(U) \subset V$” before “such that”.

• p. 61, Definition 6.5, line 4: Replace $\phi(F^{-1}(V) \cap U)$ by $\phi(U)$.

• p. 67**, Definition 6.23, display: Change $F$ to $(F|_U)$.

• p. 70, Problem 6.1(b) Hint: The identity map $\mathbb{R}' \to \mathbb{R}$.

• p. 71, line 1 of paragraph 2: Insert “usually” between “is” and “a process”.

• p. 72, line −3: “$f := f \circ \pi$” should be “$f := f \circ \pi$”.

• p. 81, Problem 7.6, line 2: $R$ should be $\mathbb{R}$.

• p. 82, Problem 7.8 (c), (d): Move the hint for (d) to the end of the hint for (c).

• p. 83, line −9: $F(k, n)$ should be $G(k, n)$.

• p. 94, Figure 8.3: The $i$ in $a_i$ should be a superscript. This occurs in two places.

• p. 105, Figure 9.4: The rightmost $\mathbb{R}^n$ should be $\mathbb{R}^m$.

• p. 106, line 5: “$S := f^{-1}(c)$” should be “$S := F^{-1}(c)$”.

• p. 109, Problem 9.10 should be starred.

• p. 112, line 5: Replace 1 by 1.

• p. 117, line −2: “$\psi(f(q)) = (y^1(f(q)), \ldots, y^n(f(q)))$” should be “$\psi(f(q)) = (y^1(f(q)), \ldots, y^m(f(q)))$”.

• p. 118, lines 1 and 3: “$\psi(f(q)) = (y^1(f(q)), \ldots, y^m(f(q)))$” should be “$\psi(f(q)) = (y^1(f(q)), \ldots, y^m(f(q)))$”.

• p. 134, line −3: Change “$M \times \mathbb{R}^n$” to “$M \times \mathbb{R}$”. In fact, in harmony with Example 12.6, one may want to change all occurrences of “$M \times \mathbb{R}$” on line −3 to “$M \times \mathbb{R}^r$”.

• p. 135, display 2: $U \times \mathbb{R}^n$ should be $U \times \mathbb{R}^r$. (“$n$” should be “$r$”.)

• p. 138, line 4: “$\mathbb{R}^n$” should be “$\mathbb{R}^r$”.

• p. 139, Problem 12.2, line 1: “about $p$” on a manifold $M$.

• p. 139, Problem 12.2 (a): “at $\phi(p)$” $\longrightarrow$ “at $\tilde{\phi}(p)$”

• p. 143, line −1: $g$ should be evaluated at $\frac{\|x\|^2 - a^2}{b^2 - a^2}$.

• p. 146, line 4, insert after $W_q$: “only finitely many of the $f_a$’s can be nonzero and”

• p. 147, Problem 13.3 (b): After “a manifold,”, insert the sentence “Assume that $A \subset U$.”

• p. 150, lines 4 and 5 in the proof of Lemma 14.1: Change “$\tilde{\phi} : TU \to U \times \mathbb{R}^n$” to “$\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$”, and “$\tilde{\phi} \circ X : U \to U \times \mathbb{R}^n$” to “$\tilde{\phi} \circ X : U \to \phi(U) \times \mathbb{R}^n$”.

• p. 152, first display: Replace $\tilde{X}(q)$ by $\tilde{X}_q$. 
• p. 160, Definition 14.14: Change “A vector field $X$ on $N$ is $F$-related to a vector field $\bar{X}$ on $M$” to “A vector field $X$ on $N$ and a vector field $\bar{X}$ on $M$ are $F$-related to each other”

• pp. 171–174: On these four pages, change “$AXA^{-1}$” to “$A^{-1}XA$”, and “$A(\cdots)A^{-1}$” to “$A^{-1}(\cdots)A$”.

• p. 172, Part (ii) of the Proof of Lemma 15.18 uses the notation from edition one. Replace it by “Apply part (i) to the matrices $A^{-1}X$ and $A$”

• p. 178, line 8: “identity” should be “identify”.

• p. 179, Problem 15.9 (b), lines 5 and 6: “elements of order 2” → “elements of order at most 2”

• p. 186, lines −4 and −3: After “If a line has rational slope…” insert “or $\infty$”.

• p. 191, line −2 of the Proof of Proposition 17.2: Apply both sides of (17.1) to $t$.

• p. 191, heading of 17.2: Change to “Local Expression for the Differential of a Function”.

• p. 191, line 4 in the proof of Lemma 17.5: Replace “$\tilde{\phi}$” by “$\tilde{\phi}$”.

• p. 197, line 9: $(V,y^1,\ldots,y^n)$ should be $(V,y^1,\ldots,y^n)$.

• p. 198, line −3: Insert “and 17.10” after “by Proposition 17.11”.

• p. 201, line −7: Both “$\mathbb{R}^n$” should be “$U$”.

• p. 202, proof of Proposition 18.3, 2nd display: “by Lemma 18.2” → “by (18.2)”

• p. 206, line 4: “for $C^\infty$ function” should be “for $C^\infty$ functions”.

• p. 207, line 4 of the Proof of Proposition 18.12: “the $C^\infty$ inverse” should be “a $C^\infty$ inverse”.

• p. 208, line 1: By Proposition 18.7(iv)

• p. 209, Problem 18.9(d): Replace the initial phrase by “As the image of a compact, connected set $G$ under a continuous map”.

• p. 214, third line of top display: Change $DD\tilde{e}$ to $DD\tilde{f}$.

• p. 215, proof of Propositions 19.7: In second line of the last display, change “(Proposition 19.5)” to “(Proposition 17.10)”. Then move Proposition 19.7 before Proposition 19.5.

• p. 216, In analogy with with the title of Subsection 17.6, change the title of Subsection 19.6 to “. . . an Immersed Submanifold”. Also change “regular” to “an immersed” on line 2 of Subsection 19.6.

• p. 218, Problem 19.3, last line: Change $i \circ c$ to $i \circ h$.

• p. 220, Problem 19.12, (c): Replace by “If $D$ is a derivation of $C^\infty(M)$ and $p \in M$, define $D_p: C^\infty_p(M) \to \mathbb{R}$ by

$$D_p[f] = (\tilde{f})'(p) \in \mathbb{R},$$

where $[f]$ is the germ of $f$ at $p$ and $\tilde{f}$ is a global extension of $f$, such as those given by Proposition 18.8. Show that $D_p[f]$ is well defined. (Hint: Apply Problem 19.7.)”

• p. 220, Problem 19.12, (d): Change “derivation” to “point-derivation”.

• p. 223, line −2 of the Proof of Proposition 20.2: “$d\left(\frac{\partial}{\partial t}|_{t_0}\omega_t\right)$” should be “$d\left(\frac{\partial}{\partial t}|_{t_0}\omega_t\right)$”.

• p. 225, (20.6) and (20.7): $(-t, p)$ in the formula should be $(-t, \varphi_t(p))$.

• p. 228, 4th line of 2nd display: Change $\sum_{i=1}^k$ to $\sum_{i=1}^\ell$. 

3
• p. 228, line 6 after the proof of Proposition 20.8: Change “Proposition 18.7(iii)⇒(i)” to “Proposition 18.7 (iv)⇒(i)”.

• p. 232, Proof of Theorem 20.12: Add to the end of the proof:

"Thus,

\[ X(\omega(Y_1, \ldots, Y_k)) = (\mathcal{L}_X \omega)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_k). \]

Solving for \((\mathcal{L}_X \omega)(Y_1, \ldots, Y_k)\) gives the formula in the theorem."

• p. 239, line 4th display: “orientation \((v_1, \ldots, v_n)\)” → “orientation \([(v_1, \ldots, v_n)]\)”.

• p. 241, line 5: Replace the sentence “But under the identification ... at \((0, 0)\)” by “Under the identification (21.1), the curve \(c(t) = (0, t)\) for \(t \in [\epsilon, \epsilon]\) maps to \(\bar{c}(t) = (1, -t)\). Hence, the tangent vector \(c'(0) = e_2\) at \(p\) maps to \(\bar{c}'(0) = -e_2\) at \(q\), and the ordered basis \(e_1, e_2\) at \(p = (0,0)\) maps to \(e_1, -e_2\) at \(q = (1,0)\).”

• p. 241, line 7 in the first paragraph: Change “Thus, at \((0,0)\)” to “Thus, at \((1,0)\)”.  

• p. 245, line 8: “\((⇒)\)” should be “\((⇐)\)”.

• p. 249, line 18: “there are” should be “there is”.

• p. 251, line 4: Change \(\rho \in U \subset S\) to \(\rho \in U \subset A\).

• p. 254, line 3 under the Subsection 22.5: “\(c((0, \epsilon]) \subset M^\circ\)” should be “\(c([0, \epsilon]) \subset M^\circ\)”.

• p. 254, second paragraph of Section 22.5: Replace the second and third sentences by

“In a coordinate neighborhood \((U, x^1, \ldots, x^n)\) in \(M\), such a vector field \(X\) can be written as a linear combination

\[ X \rho = \sum_i a_i^\rho(p) \frac{\partial}{\partial x^i} \bigg|_p \quad \text{for} \quad p \in U \cap \partial M. \]

The vector field \(X\) along \(\partial M\) is said to be smooth at \(p \in \partial M\) if there exists a coordinate chart \(U\) containing \(p\) such that the functions \(a^\rho\) on \(U \cap \partial M\) are \(C^\infty\) at \(p\); it is said to be smooth at every point \(\rho \in \partial M\).

• p. 256, 3rd line of last example: Chaneg \(T \rho C\) to \(T_{c(p)} C\).

• p. 261, display above Definition 23.1: “\(\inf_\rho L(f, P)\)” → “\(\inf_\rho U(f, P)\)”.

• p. 265, line 9: \((U, \phi)\) instead of \(\{(U, \phi)\}\). (Remove the braces.)

• p. 266, line 2: Replace “\(\phi_\alpha |_{U_\alpha \cap U_\beta}\)” and “\(\psi_\alpha |_{U_\alpha \cap U_\beta}\)” by “\(\phi_\alpha |_{U_\alpha \cap V_\beta}\)” and “\(\psi_\alpha |_{U_\alpha \cap V_\beta}\)” respectively.

• p. 267, lines 3 and 6: For consistency with equations (23.4, 23.5) on p. 264, put “det” before both occurrences of \(J\), the Jacobian.

• p. 272, Problem 23.3, line 2: “\(\Omega^k_c(M)\)” → “\(\Omega^k_c(M)\)”.

• p. 273, line 9: “smooth \(a\)” should read “a smooth”.

• p. 279, line −2: \(A^{k\times \ell}\) should be \(A^{k+\ell}\).

• p. 294, line −1: Delete one of the extra occurrences of \(\frac{\partial \phi \cdots \partial \phi}{\partial y_1 \cdots \partial y_l}\).
Example. The map $F(x,t) = \cos^2 \left( \frac{\pi}{2} t \right) x + \sin^2 \left( \frac{\pi}{2} t \right) \frac{x}{\|x\|}$ is a deformation retraction from the punctured plane $\mathbb{R}^2 - \{0\}$ to the unit circle $S^1$.

Example. The map $F$ in Example 27.6 is a deformation retraction from $\mathbb{R}^n$ to a singleton $\{p\}$.